

### A. Standard exercises:

**9.1** (Regular curve as an intersection of transversal surfaces) Let  $C \subset \Omega$  be the set of points in  $\Omega$  such that

$$f(x, y, z) = g(x, y, z) = 0,$$

where  $\Omega$  is a domain of  $\mathbb{R}^3$  and  $f, g \in C^k(\Omega)$  with  $k \geq 1$ . Assume that for a point  $(x_0, y_0, z_0) \in C$ , the matrix

$$\begin{pmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} & \frac{\partial f}{\partial z} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} & \frac{\partial g}{\partial z} \end{pmatrix}$$

has rank 2.

- (a) Show that the subsets  $\{f = 0\} \subset \mathbb{R}^3$  and  $\{g = 0\} \subset \mathbb{R}^3$  define, in a neighborhood of  $(x_0, y_0, z_0)$ ,  $C^k$  surfaces that intersect transversally (see Ex. 8.4).
- (b) Show that one can locally parametrize the set  $C$  in a neighborhood of  $(x_0, y_0, z_0)$  as a regular curve  $\gamma : I \rightarrow \Omega$  of class  $C^k$ .
- (b) Find an expression for the tangent vector  $\dot{\gamma}(t)$ .

**9.2** Recall what it means for a map  $f : \mathcal{M} \rightarrow \mathcal{N}$  between two smooth submanifolds to be smooth and what is the differential  $df_p : T_p\mathcal{M} \rightarrow T_{f(p)}\mathcal{N}$ . Prove the composition rule for derivatives: If  $f : \mathcal{M} \rightarrow \mathcal{N}$  and  $g : \mathcal{N} \rightarrow \mathcal{L}$  are smooth maps between submanifolds, then for any  $p \in \mathcal{M}$ :

$$d(g \circ f)_p = dg_{f(p)} \circ df_p.$$

**9.3** Recall that by definition a map  $f : \mathcal{M} \rightarrow \mathcal{N}$  between two smooth submanifolds is a smooth *diffeomorphism* if it is bijective and both  $f$  and  $f^{-1}$  are smooth (for the purposes of this exercise, smooth means at least  $C^1$ ).

- (a) Prove that for every  $p \in \mathcal{M}$ , the differential  $df_p : T_p\mathcal{M} \rightarrow T_{f(p)}\mathcal{N}$  is an isomorphism of vector spaces.
- (b) Deduce that there cannot exist a smooth diffeomorphism between two non-empty manifolds that do not have the same dimension.
- (c) Show, by an example, that a bijective smooth map  $f : \mathcal{M} \rightarrow \mathcal{N}$  between two smooth submanifolds is not necessarily a diffeomorphism (you may assume  $\dim(\mathcal{M}) = 1$ ).
- (\*d) Let  $f : \mathcal{M} \rightarrow \mathcal{N}$  be a smooth map. Let  $p \in \mathcal{M}$  be such that  $df_p : T_p\mathcal{M} \rightarrow T_{f(p)}\mathcal{N}$  is a bijective map. Show that  $f$  is locally a diffeomorphism around  $p$ .

**9.4** If you haven't already done so previously (since the corresponding exercises in sheet 7 and 8 were marked with an asterisk), prove that  $O(n)$  is a submanifold of  $M_n(\mathbb{R})$  and describe the tangent space  $T_I O(n)$  of this manifold at the point  $I$  (the identity matrix). Show that the exponential matrix map  $A \rightarrow \exp(A)$  maps  $T_I O(n)$  to  $SO(n)$  (use Ex. 3.7); use this map to construct a local parametrization of  $O(n)$  around  $I$  (you might want to use part (d) of the previous exercise).

**9.5** A surface is said to be *ruled* if it is a union of straight lines. More precisely, let  $\gamma : I \rightarrow \mathbb{R}^3$  be a  $C^1$  curve and  $b : I \rightarrow \mathbb{R}^3$  a  $C^1$  vector field along  $\gamma$ , then the associated ruled surface is defined by the parametrization:

$$\psi(u, v) = \gamma(u) + vb(u).$$

- (a) Give the necessary and sufficient conditions for such a ruled surface to be locally a regular surface (that is, for the map  $\psi$  to be an immersion).
- (b) Let  $C$  be a curve in  $\mathbb{R}^3$ . A *cone* with vertex  $q \in \mathbb{R}^3$  and base  $C$  is the union of lines passing through  $q$  and a point of  $C$ . Give the necessary and sufficient conditions for a cone to be a regular surface in a neighborhood of its base. Then give an explicit parametrization of this cone.
- (c) Explain what a Möbius strip is and provide a parametrization of this surface as a ruled surface in  $\mathbb{R}^3$ .

**9.6** Show that the one-sheeted hyperboloid  $H$  given by  $x^2 + y^2 - z^2 = 1$  is a doubly ruled surface (i.e., ruled in two different ways), and then give a regular parametrization of this surface based on one of these rulings.

*Hint:* Write the equation as  $x^2 - 1 = z^2 - y^2$  and factorize. Deduce algebraically the equation of a line contained in  $H$ , then parametrize it and rotate it around the  $Oz$ -axis.

## B. Bonus exercise:

**9.7** In this exercise, we construct an example of an injective immersion that is not an embedding. The *lemniscate of Gerono* is the plane curve defined by the equation

$$4x^2 - 4y^2 - x^4 = 0,$$

that is, the set

$$C = \{(x, y) \in \mathbb{R}^2 \mid 4x^2 - 4y^2 - x^4 = 0\}.$$

- (a) Show that  $C$  is not a differentiable submanifold of  $\mathbb{R}^2$ .

- (b) Is the restriction of this curve to  $\mathbb{R}^2 \setminus \{(0, 0)\}$  a differentiable submanifold?  
(c) Verify that  $\gamma : \left(-\frac{\pi}{2}, \frac{3\pi}{2}\right) \rightarrow \mathbb{R}^2$  defined by

$$\gamma(t) = (2 \cos t, \sin(2t))$$

is a regular parametrization of  $C$ . More precisely, prove that:

- (i)  $\gamma$  is an immersion of the open interval  $\left(-\frac{\pi}{2}, \frac{3\pi}{2}\right)$  into the plane.  
(ii)  $\gamma$  is injective.  
(iii)  $\gamma$  defines a bijection between the open interval  $\left(-\frac{\pi}{2}, \frac{3\pi}{2}\right)$  and the curve  $C$ .  
(iv) Explain what happens to  $\gamma$  as  $t \rightarrow -\frac{\pi}{2}$  and  $t \rightarrow \frac{3\pi}{2}$ .  
(v) Prove that  $\gamma$  is not an embedding of the interval  $\left(-\frac{\pi}{2}, \frac{3\pi}{2}\right)$  into the plane (that is, it is not a homeomorphism onto its image).